On Automorphisms of Matrix Invariants Induced From the Trace Ring

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ABSTRACT

1. INTRODUCTION

In this paper we continue the study of the automorphism group of the space $Q_{m,n}$ initiated in [14]. Recall that $Q_{m,n}$ is the algebraic quotient space for the simultaneous conjugation action of $\mathrm{PGL}_n(k)$ on the affine space $M_n(k)^m$ of m-tuples of $n \times n$ matrices. Here and throughout the paper the base field k will be assumed to be algebraically closed and of characteristic 0.

Recall that the affine variety $Q_{m,n}$ is stratified according to representation type. The representation type of a point $p \in Q_{m,n}$ is defined as follows. Let $(X_1, \dots, X_m) \in M_n(k)$ be an element of the unique closed PGL_n orbit which projects to p under the quotient map. Then the representation

$$\phi : k\{u_1, \dots, u_m\} \to M_n(k),$$
$$u_i \to X_i \tag{1}$$

of the free algebra $F_m = k\{u_1, \dots, u_m\}$ is semisimple; see [1]. We say that p has representation type $\tau = (e_1, d_1; \cdots ; e_r, d_r)$ if ϕ is the sum of r irreducible representations ϕ_i of dimension d_i and multiplicity e_i. The set of all points of representation type τ will be denoted by $Q_{m,n}(\tau)$. The group of algebraic representation-type-preserving automorphisms of $Q_{m,n}$ will be denoted by $\mathbf{Aut}(Q_{m,n})$.

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In [14] we proved the following result.

THEOREM 1.1. *Assume* $m \geqslant n + 1$. *Then*

(a) [14, *Theorem* 1.2] $\mathbf{Aut}(Q_{m,n})$ *acts transitively on* $Q_{m,n}(\tau)$ *for any representation type* τ.

(b) [14, *Theorem* 1.3] $\mathbf{Aut}(Q_{m,n})$ *acts s-transitively on* $Q_{m,n}(1, n)$ *for any positive integer* s.

The automorphisms of $Q_{m,n}$ which appear in the proof of Theorem 1.1 are all induced by automorphisms of the free polynomial algebra $F_m = k\{u_1, \ldots, u_m\}$. In this paper we study the broader class of automorphisms of $Q_{m,n}$ induced by k-algebra automorphisms of the trace ring $T_{m,n}$. For generalities on such automorphisms see Section 3. These new automorphisms allows us to strengthen Theorem 1.1 as follows.

THEOREM 1.2. *Assume* $m \geqslant n + 1$. *Then for any positive integer* s *the group* $\mathbf{Aut}(Q_{m,n})$ *acts s-transitively on the stratified space* $Q_{m,n}$. *In other words, suppose* $(p_1, \ldots, p_s)$ *and* $(q_1, \ldots, q_s)$ *are two s-tuples of distinct points in* $Q_{m,n}$ *such that* $\tau(p_i) = \tau(q_i)$ *for* $i = 1, \ldots, s$. *Then there exists a* $g \in \mathbf{Aut}(Q_{m,n})$ *such that* $g(p_i) = q_i$ *for* $i = 1, \ldots, s$.

We give a proof of this theorem in Section 5. The condition $m \geqslant n + 1$ is necessary in the case $n = 1$, since every automorphism of $Q_{1,1} = k^1$ is of the form $x \to ax + b$. We do not know whether or not the theorem holds for $m \leqslant n$ when $n \geqslant 2$. We also note that for some choices of $\{p_i\}, \{q_i\}$ the element g in the statement of Theorem 1.2 cannot be induced by an automorphism of the polynomial algebra F_m; see Remark 3.4.

As we mentioned above, an automorphism of F_m induces an automorphism of $T_{m,n}$ which in turn induces a representation type preserving automorphism of $Q_{m,n}$; see Section 3 for details. In Sections 6 and 7 we study the resulting diagram of group homomorphisms

$$\mathrm{Aut}(F_m) \to \mathrm{Aut}(T_{m,n}) \to \mathbf{Aut}(Q_{m,n}).$$

The main result of these sections is that the second map is not surjective. In other words,

THEOREM 1.3. *Suppose* $n \geqslant 3$ *and* $m \geqslant 2$. *Then there exists a* $g \in \mathbf{Aut}(Q_{m,n})$ *which is not induced by an automorphism of the trace ring* $T_{m,n}$.

The automorphism g in the theorem is, in fact, of a very simple form; see Proposition 7.2. The key part of our proof of Theorem 1.3 is the criterion for $g \in \mathrm{Aut}(T_{m,n})$ to be trivial given by Corollary 6.2. This criterion is closely related to a theorem of Guralnick and Montgomery [5]. In Section 6 we present a proof of this theorem based on our criterion.

2. PRELIMINARIES

The Trace Ring $T_{m,n}$

Let $x_{ij}^{(l)}$ be mn^2 independent commuting variables. Here $i, j = 1, \ldots, n$ and $l = 1, \ldots, m$. The ring of generic matrices $G_{m,n}$ is the k-subalgebra of the matrix algebra $M_n(k[x_{ij}^{(l)}])$ generated by the m elements $X_1 = (x_{ij}^{(1)}), \ldots, X_m = (x_{ij}^{(m)})$. The trace ring $T_{m,n}$ is defined as the subalgebra of $M_n(k[x_{ij}^{(l)}])$ generated by elements of $G_{m,n}$ and the coefficients of their characteristic polynomials. The latter are viewed as scalar matrices in $M_n(k[x_{ij}^{(l)}])$. In fact, since our base field k is assumed to be of characteristic 0, Newton's formulas say that $T_{m,n}$ is generated by elements of $G_{m,n}$ and their traces. Denote the subalgebra of scalar elements of $T_{m,n}$ by $C_{m,n}$, i.e.

$$C_{m,n} = k\left[x_{ij}^{(l)}\right] I_{n\times n} \cap T_{m,n}.$$

In other words, $C_{m,n}$ is the k-algebra generated by elements of the form $\operatorname{tr} Z$ for $Z \in G_{m,n}$.

Note that the natural $(\mathrm{Z}^+)^m$-grading of $G_{m,n}$ by the multidegree in $(X_1, \ldots, X_m)$ extends to a $(\mathrm{Z}^+)^m$-grading of $T_{m,n}$. We shall denote the degree of an element $Z \in T_{m,n}$ with respect to X_i by $\deg_i Z$ and the total degree of Z by $\deg Z$.

The Universal Property of $T_{m,n}$

As a direct consequence of the above definition, the trace ring $T_{m,n}$ has the following universal property.

PROPOSITION 2.1. *Let R be a commutative k-algebra, N be a subalgebra of the matrix algebra $M_n(R)$ such that* $\operatorname{tr} N \subset N$, *and $a_1, \ldots, a_m \in N$. Then there exists a unique k-algebra homomorphism $h : T_{m,n} \to N$ such that $h \circ \operatorname{tr} = \operatorname{tr} \circ h$ and $h(X_l) = a_l$ for $l = 1, \ldots, m$.*

Indeed, we obtain h by specializing each $x_{ij}^{(l)}$ to the (i, j) entry of a_l.

We remark that one can prove a similar statement about any algebra N with trace by using the embedding theorem of Procesi [12]; see [14, Lemma 2.8]. However, in the sequel we will only appeal to the elementary version of this result stated above. As a corollary we obtain the following fact.

COROLLARY 2.2. *The natural embedding $G_{m-1,n} \hookrightarrow G_{m,n}$ which takes X_l to X_l for $l = 1, \ldots, m-1$ extends to an embedding $T_{m-1,n} \hookrightarrow T_{m,n}$.*

Proof. Applying Proposition 2.1 to the $m-1$-tuple $X_1, \ldots, X_{m-1} \in T_{m,n}$, we obtain a homomorphism $h : T_{m-1,n} \to T_{m,n}$. To see that h is injective, we apply Proposition 2.1 once again. It says that there is a homomorphism $g: T_{m,n} \to T_{m-1,n}$ such that $g(X_l) = X_l$ for $l = 1, \dots, m-1$ and $g(X_m) = 0$. Then $g \circ h(X_l) = X_l$. Moreover, $g \circ h: T_{m-1,n} \to T_{m-1,n}$ commutes with the trace function. Hence, $g \circ h$ is the identity map on $T_{m-1,n}$. In particular, h is injective. ■

REMARK 2.3. In the sequel we shall identify $T_{m-1,n}$ with its image in $T_{m,n}$.

Properties of $T_{m,n}$ and $C_{m,n}$

We shall make use of the following properties of $T_{m,n}$ and $C_{m,n}$.

PROPOSITION 2.4. *Any nonzero element $Z \in T_{m,n}$ is nonsingular, i.e.,* $\det Z \neq 0$. *In particular, $T_{m,n}$ is a domain.*

This is a consequence of the fact that $G_{m,n}$ is a domain and $T_{m,n}$ embeds in its total field of fractions, which is a division algebra. See [2, Section 12.6] for details.

PROPOSITION 2.5 (Procesi [11]). *$C_{m,n}$ is the center of $T_{m,n}$.*

PROPOSITION 2.6 (Sibirskii [16], Procesi [11]). *$C_{m,n}$ is the ring of invariants for the conjugation action of* $\mathrm{PGL}_n(k)$ *on $M_n(k)^m$. In other words, $C_{m,n}$ is the coordinate ring of the affine variety $Q_{m,n}$.*

PROPOSITION 2.7 (Shirshov [15, Section 4.2]). *$T_{m,n}$ is finitely generated as $C_{m,n}$-module.*

We refer the reader to [4], [6], and [13] for a more detailed account of the properties of the trace ring.

Automorphisms of $T_{m,n}$ Commute with tr

In the sequel the characteristic polynomial of a matrix M will be denoted by

$$\mathrm{char}_M(t) = t^n + \sum_{i=0}^{n-1} (-1)^i c_i(M) t^{n-i}.$$

The group of k-algebra automorphisms of $T_{m,n}$ will be denoted by $\mathrm{Aut}(T_{m,n})$

PROPOSITION 2.8. *Let f be a k-algebra automorphism of $T_{m,n}$. Then for any $Z \in T_{m,n}$ and any $i = 0, 1, \dots, n-1$ we have $c_i(f(Z)) = f(c_i(Z))$. In particular,* $\operatorname{tr} f(Z) = f(\operatorname{tr} Z)$ *and* $\det f(Z) = f(\det Z)$.

Proof. By the Cayley-Hamilton identity an element $Z \in T$ satisfies two monic polynomials of degree n:

$$\operatorname{char}_Z(t) = t^n + \sum_{i=0}^{n-1} (-1)^i c_i(Z) t^{n-i},$$

$$f^{-1} \operatorname{char}_{f(Z)}(t) = t^n + \sum_{i=0}^{n-1} (-1)^i f^{-1}[c_i(f(Z))] t^{n-i}.$$

If Z has distinct eigenvalues, then these polynomials must coincide. That is, $c_i(Z) = f^{-1} c_i(f(Z))$, as desired. Now let Z be an arbitrary element of $T_{m,n}$. Note that X_1 has distinct eigenvalues, and hence so does $Z_s = Z + sX_1$ for all but finitely many $s \in k$. Then $c_i(Z_s) = f^{-1}(c_i(f(Z_s)))$ holds for all but finitely many $s \in k$; hence, it holds for all $s \in k$, and in particular, for $s = 0$. ■

COROLLARY 2.9. *Let f be a k-algebra automorphism of $T_{m,n}$. Then f is trivial if and only if $f(X_i) = X_i$ for $i = 1, \ldots, m$.*

Proof. Suppose $f(X_i) = X_i$ for $i = 1, \ldots, m$. Then $f(Z) = Z$ for every $Z \in G_{m,n}$. By Proposition 2.8 $f(c_i(Z)) = c_i(f(Z)) = c_i(Z)$ for every $Z \in G_{m,n}$ and every $i = 0, 1, \ldots, n-1$. Since $T_{m,n}$ is generated by elements of the form Z and $c_i(Z)$ as Z ranges over $G_{m,n}$, the automorphism f is trivial on all of $T_{m,n}$. ■

PROPOSITION 2.10. *Let Λ be the collection of all m-tuples $Y_1, \ldots, Y_m \in T_{m,n}$ which generate $T_{m,n}$ as a k-algebra with trace. Then the correspondence*

$$\operatorname{Aut}(T_{m,n}) \to \Lambda,$$

$$f \to (f(X_1), \ldots, f(X_m))$$

is a bijection.

Proof. Injectivity follows from Corollary 2.9. To prove surjectivity suppose $Y_1, \ldots, Y_m$ generate $T_{m,n}$ as an algebra with trace. Then by Proposition 2.1 there exists an endomorphism $f : T_{m,n} \to T_{m,n}$ which takes $X_1, \ldots, X_m$ to $Y_1, \ldots, Y_m$ respectively. Since f commutes with tr, it is surjective. It remains to show that f is also injective. Assume the contrary. Then

$$(0) \subset \operatorname{Ker} f \subset \operatorname{Ker} f^2 \subset \cdots$$

is a strictly increasing sequence of $C_{m,n}$-submodules of $T_{m,n}$ contradicting Proposition 2.7. ∎

Representations

Let M be a k-algebra. A *representation* (an *irreducible representation*) of F_m in M is a k-algebra homomorphism (a surjective k-algebra homomorphism)

$$\rho : F_m \to M.$$

Two representations $\rho_1 : F_m \to M_1$ and $\rho_2 : F_m \to M_2$ are said to be *equivalent* if there exists an isomorphism $f : M_1 \to M_2$ such that $\rho_2 = f \circ \rho_1$. We write $\rho_1 \approx \rho_2$.

A representation ϕ is called *semisimple* if its image is a finite-dimensional semisimple k-algebra, i.e. a finite direct sum of matrix algebras over k. A semisimple representation can thus be viewed as a direct sum of irreducible representations in matrix algebras over k. Grouping the equivalent ones together, we write

$$\phi = \phi_1^{e_1} \oplus \cdots \oplus \phi_r^{e_r},$$

where $\phi_i : F_m \to M_{d_i}(k)$ is irreducible for $i = 1, \dots, r$ and $\phi_i \not\approx \phi_j$ for $i \neq j$. As we mentioned in the introduction, every point $p \in Q_{m,n}$ is represented by an m-tuple of $n \times n$ matrices $(X_1, \dots, X_m)$ such that the representation

$$\rho : F_m \to M_n(k),$$

$$u_l \to X_l$$

for $l = 1, \dots, m$ is semisimple. Up to equivalence ρ is unique; we shall refer to it as the semisimple representation associated to p. Suppose

$$\rho = \rho_1^{e_1} \oplus \cdots \oplus \rho_r^{e_r}, \tag{2}$$

where $\rho_i : F_m \to M_{d_i}(k)$ is irreducible for $i = 1, \dots, r$ and $\rho_i \not\approx \rho_j$ for $i \neq j$.

Then we define the representation type of p (or ρ) to be the unordered r-tuple of pairs (e_j, d_j) and write

$$\tau(p) = (e_1, d_1; \cdots; e_r, d_r).$$

Note that $\sum_{i=1}^{r} e_i d_i = n$.

If the semisimple representation associated to p is as in (2), then we define the reduced semisimple representation

$$\phi : F_m \to M_{d_1} \times \cdots \times M_{d_r}$$

associated to p by $\phi = \rho_1 \oplus \cdots \oplus \rho_r$.

PROPOSITION 2.11 [1, Theorem 9.2; 14, Lemma 2.4]. *For $i = 1, \ldots, r$ let ϕ_i be representation of F_m in a matrix algebra over k. Then $\rho = \rho_1 \oplus \cdots \oplus \rho_r$ is irreducible if and only if $\rho_1, \ldots, \rho_r$ are irreducible and pairwise inequivalent.*

In particular, the reduced semisimple representation associated to $p \in Q_{m,n}$ is always irreducible.

3. WELL-BEHAVED s-TUPLES

The automorphisms of $Q_{m,n}$ which will be used in our proof of Theorem 1.2 will all be induced from automorphisms of $T_{m,n}$ in the following simple way. Let f be a k-algebra automorphism f of $T_{m,n}$. Since $C_{m,n}$ is the center of $T_{m,n}$ (Proposition 2.5), f induces an automorphism of $C_{m,n}$. We denote the dual automorphism of $Q_{m,n} = \operatorname{Spec} C_{m,n}$ by f_*; see Proposition 2.6.

It is easy to see that f_* can also be described as follows. Consider the automorphism f^* of $k^{mn^2} = M_n(k)^m$ (as an affine variety) given by

$$A = (A_1, \ldots, A_m) \to (f(X_1)(A), \ldots, f(X_m)(A)).$$

This automorphism commutes with the conjugation action of PGL_n on $M_n(k)^m$. Hence, it descends to an automorphism f_0 of the quotient variety $Q_{m,n}$; see [10, Theorem 1.1; 14, Lemma 2.2].

We claim that $f_* = f_0$. Indeed, suppose $p \in Q_{m,n}$ is represented by $(A_1, \ldots, A_m) \in M_n(k)^m$. It is enough to check that $f(c)(p) = c(f_0(p))$ for every c in $C_{m,n}$. Both sides are k-algebra homomorphisms $C_{m,n} \to k$. Thus we may assume without loss of generality that $c = \operatorname{tr}(X_{i_1} \cdots X_{i_N})$ for some $1 \leqslant i_1, \ldots, i_N \leqslant m$. Then by Proposition 2.8

$$f(c)(p) = \operatorname{tr} f(X_{i_1} \cdots X_{i_N})(p) = \operatorname{tr} f^*(A_{i_1} \cdots A_{i_N}) = c(f_0(p)),$$

as desired.

Let σ be an automorphism of the free algebra $F_m = k\{u_1, \ldots, u_m\}$. For $j = 1, \ldots, m$ let Y_j be the image of $\sigma(u_i)$ in $T_{m,n}$ under the map $F_m \to T_{m,n}$ which takes u_i to X_i. Then the k-algebra generated by $Y_1, \ldots, Y_m$ contains $X_1, \ldots, X_m$. Hence, $Y_1, \ldots, Y_m$ generate $T_{m,n}$ as an algebra with trace. By Proposition 2.10 there is a unique k-algebra automorphism of $T_{m,n}$ which takes X_i to Y_i for $i = 1, \ldots, m$. We denote this automorphism by σ'. Note that by our description of f_* above, σ'_* is exactly the automorphism of $Q_{m,n}$ which we denoted by σ_* in [14].

LEMMA 3.1. *Let f be a k-algebra automorphism of $T_{m,n}$. Then f_* is a representation-type-preserving automorphism of $Q_{m,n}$.*

Proof. The argument of the proof of [14, Proposition 4.1] goes through unchanged in this case. We briefly recall it here. Let p be a point of $Q_{m,n}$ represented by a semisimple m-tuple of matrices $A = (A_1, \ldots, A_m)$. Then by the definition of partial order on the representation types we have $\tau(A) \geqslant \tau(f^*(A))$, i.e. $\tau(p) \geqslant \tau(f_*(p))$. Applying the same argument to f^{-1}, we obtain equality. ■

We thus have a sequence of group homomorphisms

$$\operatorname{Aut}(F_m) \to \operatorname{Aut}(T_{m,n}) \to \mathbf{Aut}(Q_{m,n})$$

where the first map is given by $\sigma \to \sigma'$ and the second map is given by $f \to f_*$. We shall return this sequence in Section 7.

We now turn our attention to Theorem 1.2.

DEFINITION 3.2. We call an s-tuple $p_1, \ldots, p_s \in Q_{m,n}$ *well behaved* if no two of the associated semisimple representations $\rho_1, \ldots, \rho_s : F_m \to M_n(k)$ have an isomorphic irreducible component.

PROPOSITION 3.3. *The assertion of Theorem* 1.2 *holds if* $(p_1, \ldots, p_s)$ *and* $(q_1, \ldots, q_s)$ *are well-behaved s-tuples. Moreover, in this case there is a* $\sigma \in \operatorname{Aut}(F_m)$ *such that* $\sigma'_*(p_i) = q_i$ *for* $i = 1, \ldots, s$.

Proof. Let $\rho_i, \mu_i : F_m \to M_n(k)$ be the semisimple representations associated to p_i and q_i respectively. Denote the representation type of p_i by $\tau = (e_{i1}, d_{i1}; \dots ; e_{ir_i}, d_{ir_i})$; by our assumption τ is also the representation type of q_i. Write

$$\rho_i = \rho_{i1}^{e_{i1}} \oplus \cdots \oplus \rho_{ir_i}^{e_{ir_i}} \quad \text{and} \quad \mu_i = \mu_{i1}^{e_{i1}} \oplus \cdots \oplus \mu_{ir_i}^{e_{ir_i}}, \tag{3}$$

where $\rho_{ij}, \mu_{ij} : F_m \to M_{d_{ij}}(k)$ are irreducible representations and $\rho_{ij} \not\approx \rho_{il}, \mu_{ij} \not\approx \mu_{il}$ for $j \neq l$.

By our assumption the representations ρ_{ij} are pairwise inequivalent and the representations μ_{ij} are pairwise inequivalent. Hence, the representations

$$\rho = \bigoplus_{i,j} \rho_{ij} \quad \text{and} \quad \mu = \bigoplus_{ij} \mu_{ij} : F_m \to \bigoplus_{i,j} M_{d_{ij}}(k)$$

have the same representation type; namely $(1, d_{11}; \cdots 1, d_{sr_s})$. By Proposition 2.11 ρ and μ are irreducible. Since each $d_{ij} \leqslant n \leqslant m + 1$, we can appeal to [14, Theorem 4.4], which says that $\rho_{ij} \circ \sigma = \mu_{ij}$ for some $\sigma \in \mathrm{Aut}(F_m)$. Therefore, $\rho_i \circ \sigma = \mu_i$ and hence $\sigma'_*(p_i) = q_i$ for $i = 1, \dots, s$. ∎

REMARK 3.4. Note that the condition that the s-tuples $(p_1, \dots, p_s)$ and $(q_1, \dots, q_s)$ are well behaved is essential here. Indeed, for any $\sigma \in \mathrm{Aut}(F_m)$, the s-tuple $(\sigma_* p_1, \dots, \sigma_* p_s)$ is well behaved if and only if $(p_1, \dots, p_s)$ is well behaved. This follows from the fact that representations $\rho, \mu : F_m \to M$ are equivalent if and only if $\rho \circ \sigma$ and $\mu \circ \sigma$ are equivalent. In particular, an element of $\mathbf{Aut}(Q_{m,n})$ of the form σ'_* cannot carry a well-behaved s-tuple into one which is not well behaved. As we shall see in Section 5, we can get around this difficulty by considering elements of $\mathbf{Aut}(Q_{m,n})$ of the form f_* for $f \in \mathrm{Aut}(T_{m,n})$.

4. TAME AUTOMORPHISMS

In this section we discuss some properties of the group $\mathrm{TAut}(T_{m,n})$ of tame automorphisms of $T_{m,n}$. We shall view $\mathrm{TAut}(T_{m,n})$ as an infinite-dimensional algebraic group as defined by Shafarevich in [17]. We will show that this group is covered by an ascending collection $\{G_d\}$ of irreducible finite-dimensional varieties. For this reason, $\mathrm{TAut}(T_{m,n})$ behaves in some instances as an irreducible linear algebraic group. The results of this section will be used in the proof of Theorem 1.2 in Section 5.

Let σ be an automorphism of the free polynomial algebra $F_m = k\{u_1, \ldots, u_m\}$. Recall that σ is linear if it is given by

$$g(u_i) = \sum_{j=1} a_{ij} u_j,$$

where $a_{ij} \in k$ and the matrix (a_{ij}) is nonsingular. The automorphism σ is lower triangular if for every $i = 1, \ldots, m$ there exists a polynomial P_i in $i - 1$ variables such that $\sigma(u_i) = u_i + P_i(u_1, \ldots, u_{i-1})$.

Similarly, an automorphism f of $T_{m,n}$ is linear if

$$f(X_i) = \sum_{j=1} a_{ij} X_j$$

where $a_{ij} \in k$ and the matrix (a_{ij}) is nonsingular. An automorphism of $T_{m,n}$ is lower triangular if it is of the form

$$\begin{aligned} X_1 &\to X_1, \\ X_2 &\to Y_1 + X_2, \\ &\vdots \\ X_m &\to Y_{m-1} + X_m, \end{aligned} \tag{4}$$

where Y_i is an element of $T_{i,n}$. Here we view $T_{i,n}$ as a subring of $T_{m,n}$ for $i = 1, \ldots, m - 1$; see Remark 2.3.

An automorphism of F_m [or $T_{m,n}$] is called *tame* if it can be written as a composition of linear and lower triangular automorphisms. Tame automorphisms form a subgroup of $\mathrm{Aut}(F_m)$ [$\mathrm{Aut}(T_{m,n})$], which is denoted by $\mathrm{TAut}(F_m)$ [$\mathrm{TAut}(T_{m,n})$]. Conjecturally $\mathrm{TAut}(F_m) = \mathrm{Aut}(F_m)$ and $\mathrm{TAut}(T_{m,n}) = \mathrm{Aut}(T_{m,n})$.

As a direct consequence of our definitions we obtain the following lemma.

LEMMA 4.1. *Let σ be a tame automorphism of F_m. Then σ' is a tame automorphism of $T_{m,n}$.*

Let $G(d) = \{h_1 g_1 \cdots h_d g_d\}$, where each h_j is linear and each g_i is lower triangular of degree $\leqslant d$. Here by the degree of an automorphism f of $T_{m,n}$ we mean the maximum of the total degrees of $g(X_l)$ as $l = 1, \ldots, m$; see Section 2. Clearly $\mathrm{TAut}(T_{m,n}) = \bigcup_{d \geqslant 1} G_d$.

Note that an element of G_d is of degree at most d^d. In other words, if $Z_1, \ldots, Z_N$ is a basis of the k-vector space of elements of $T_{m,n}$ of degree $\leqslant d^d$, then

$$f(X_i) = \sum_{j=1}^{N} c_{ij} Z_j \tag{5}$$

for some $c_{ij} \in k$. The scalars c_{ij} can be thought of as the coordinates of f. Since an element $f \in G_d$ is uniquely determined by its mN coordinates c_{ij}, we can think of G_d as a subset of the affine space k^{mN}.

LEMMA 4.2. *G_d is an irreducible constructible subset of k^{mN}.*

Proof. Let L_d be the set of lower triangular automorphisms of $T_{m,n}$ of degree $\leqslant d$. Since each Y_i in (4) ranges over the finite-dimensional k-vector space of elements of $T_{i,n}$ of degree $\leqslant d$, L_d is isomorphic to an affine space. Thus we have a regular surjective map

$$(\mathrm{GL}_m(k) \times L_d)^d \to G_d$$

given by $(h_1, g_1, \ldots, h_d, g_d) \to h_1 g_1 \cdots h_d g_d$. Its image is constructible by Chevalley's theorem. Since $\mathrm{GL}_m(k) \times L_d$ is irreducible, so is G_d. ∎

LEMMA 4.3. *Let $p, q \in Q_{m,n}$, and let W be the set of all $f \in G_d$ such that $(f_* p, f_* q)$ is well behaved. Then W is Zariski-open in G_d.*

Proof. Suppose the semisimple representations associated to p and q are

$$\rho = \rho_1^{a_1} \oplus \cdots \oplus \rho_r^{a_r} \quad \text{and} \quad \mu = \mu_1^{b_1} \oplus \cdots \oplus \mu_s^{b_s},$$

where $\rho_i, \mu_i : F_m \to M_{d_i}(k)$ are irreducible.

Let $f \in G_d$ be as in (5). Then the semisimple representations associated to $f_* p$ and $f_* q$ are of the form

$$\rho(c) = \rho_1(c)^{a_1} \oplus \cdots \oplus \rho_r(c)^{a_r} \quad \text{and} \quad \mu(c) = \mu_1(c)^{b_1} \oplus \cdots \oplus \mu_s(c)^{b_s},$$

where $\rho_i(c), \mu_i(c)$ are representations of F_m in $M_{d_i}(k)$ which depend on $c = (c_{ij})$.

Fixing p and q and letting f vary over G_d, we see that for every $P \in F_m$ the entries of $\rho_i(c)(P)$ and $\mu_i(c)(P)$ are polynomials in c_{ij}. Moreover, for $f \in G_d$ the representation type of $f_* p$ is the same as the representation type of p; see Lemma 3.1. Hence, the representations $\rho_i(c) : F_m \to M_{d_i}(k)$ are irreducible. Similarly each $\mu_i(c)$ is irreducible.

Therefore, $(f_* p, f_* q)$ is not well behaved iff for some $1 \leqslant \alpha \leqslant r$ and $1 \leqslant \beta \leqslant s$ the representations $\rho_\alpha(c)$ and $\mu_\beta(c)$ are equivalent. Since they are irreducible, this happens if and only if

$$\operatorname{tr} \rho_\alpha(c)(P) = \operatorname{tr} \mu_\beta(c)(P)$$

for every $P \in F_m$.

These are polynomial conditions on c_{ij}. They cut out a closed subset of G_d for every α and β. The set W is the complement of the union of these closed sets taken over all pairs (α, β) with $\alpha = 1, \dots, r$ and $\beta = 1, \dots, s$. Therefore, W is Zariski-open in G_d. ■

Recall that a representation ρ of F_m is *saturated* if its restriction to $F_{m-1} = k\{u_1, \dots, u_{m-1}\}$ is irreducible. A point $p \in Q_{m,n}$ is saturated if its reduced semisimple representation is saturated. In other words, p is saturated if its associated semisimple representation ρ has the property that $\rho(F_{m-1}) = \rho(F_m)$.

LEMMA 4.4. *Let $p \in Q_{m,n}$. Then the set*

$$\{f \in G_d : f_* p \text{ is saturated}\}$$

is Zariski-open in G_d.

Proof. Denote the semisimple representations associated to p by

$$\rho = \rho_1^{e_1} \oplus \cdots \oplus p_r^{e_r},$$

where $\rho_i : F_m \to M_{d_i}(k)$ are inequivalent irreducible representations.

Let $f \in G_d$ be as in (5). As we saw in the proof of the previous lemma, the semisimple representation associated to $f_* p$ is of the form

$$\rho(c) = \rho_1(c)^{e_1} \oplus \cdots \oplus \rho_r(c)^{e_r}.$$

Here the representations $\rho_1(c), \dots, \rho_r(c)$ are inequivalent and irreducible, and the entries of the matrix $\rho_i(c)(P)$ are polynomials in c_{ij} for every

$P \in F_m$ and every $i = 1, \ldots, m$. Thus $f_* p$ is saturated whenever

$$\rho_1(c) \oplus \cdots \oplus \rho_r(c)(u_1), \ldots, \rho_1(c) \oplus \cdots \oplus \rho_r(c)(u_{m-1})$$

generate $M_{d_1} \oplus \cdots \oplus M_{d_r}$ as a k-algebra. This is an open condition on c_{ij}. ∎

5. PROOF OF THEOREM 1.2

In this section we complete the proof of Theorem 1.2. In view of Proposition 3.3, we only need to prove the following proposition.

PROPOSITION 5.1. *Assume $m \geqslant n + 1$. Then for any s-tuple $(p_1, \ldots, p_s)$ of distinct points of $Q_{m,n}$ there exists a $g \in$ **Aut**$(Q_{m,n})$ such that the s-tuple $(gp_1, \ldots, gp_s)$ is well behaved. Moreover, we can take $g = f_*$ for some $f \in$ TAut$(T_{m,n})$.*

To see how Theorem 1.2 follows from Proposition 3.3 and Proposition 5.1, let $(p_1, \ldots, p_s)$ and $(q_1, \ldots, q_s)$ be as in the statement of Theorem 1.2. By Proposition 5.1 we can find g and $\tilde{g} \in$ **Aut**$(Q_{m,n})$ such that the s-tuples $(gp_1, \ldots, gp_s)$ and $(\tilde{g}q_1, \ldots, \tilde{g}q_s)$ are both well behaved. Hence, by Proposition 3.3 there exists an $h \in$ **Aut**$(Q_{m,n})$ such that $hgp_i = \tilde{g}q_i$ for $i = 1, \ldots, s$. In other words, $\tilde{g}^{-1}hg$ takes p_i to q_i for each i, as desired.

We now turn to the proof of Proposition 5.1.

REDUCTION 1. *We may assume without loss of generality that $s = 2$.*

Proof. Suppose for each $i \neq j$ there exists an $f_{ij} \in$ TAut$(T_{m,n})$ such that $(f_* p_i, f_* p_j)$ is well behaved. Since

$$\mathrm{TAut}(T_{m,n}) = \bigcup_d G_d,$$

we can find a single d such that G_d contains every f_{ij}. For this d let W be the set of all $f \in G_d$ such that $(f_* p_1, \ldots, f_* p_s)$ is well behaved, and let W_{ij} be the set of all $f \in G_d$ such that $(f_* p_i, f_* p_j)$ is well behaved. Note that

$$W = \bigcap_{1 \leqslant i < j \leqslant s} W_{ij};$$

see Definition 3.2. Each W_{ij} is nonempty by our choice of d and is open in G_d by Lemma 4.3. Since G_d is irreducible (Lemma 4.2), this implies that W is nonempty. ∎

From now on we shall assume that $s = 2$ and denote p_1 and p_2 by p and q respectively.

REDUCTION 2. *We may assume without loss of generality that p and q are saturated.*

Proof. It is enough to show that there exists an $f \in \mathrm{TAut}(T_{m,n})$ such that $f_* p$ and $f_* q$ are saturated; we can then replace (p, q) by $(f_* p, f_* q)$.

In fact, we only need to show that there exists an $f \in \mathrm{TAut}(T_{m,n})$ such that $f_* p$ is saturated. Indeed, let W_p be the set of all f such that $f_* p$ is saturated, and let W_q be the set of all f such that $f_* q$ is saturated. We want to show that $W_p \cap W_q \neq \varnothing$. Indeed, if $W_p \neq \varnothing$ and $W_q \neq \varnothing$, then for d sufficiently large $W_p \cap G_d \neq \varnothing$ and $W_q \cap G_d \neq \varnothing$. By Lemma 4.4 both $W_p \cap G_d$ and $W_q \cap G_d$ are Zariski-open in G_d. Since G_d is irreducible (Lemma 4.2), their intersection is nonempty. Hence, $W_p \cap W_q \neq \varnothing$, as desired.

Now let ϕ be the reduced semisimple representation associated to p. By [14, 8.1, 8.3 and 3.4] there exists a tame automorphism σ of F_m such that $\phi \circ \sigma$ is saturated. This means that $(\sigma')_* p$ is saturated. ■

Note that the last argument makes use of the assumption $m \geqslant n + 1$. From now on we shall assume that p and q are saturated. Let the semisimple representations associated to p and q be

$$\rho = \rho_1^{a_r} \oplus \cdots \oplus \rho_r^{a_r} \quad \text{and} \quad \mu = \mu_1^{b_1} \oplus \cdots \oplus \mu_s^{b_s}$$

respectively. Here $\rho_i, \mu_i : F_m \to M_{d_i}(k)$ are irreducible representations; $\rho_i \not\simeq \rho_j$, $\mu_i \not\simeq \mu_j$ for $i \neq j$.

Our further arguments rely on the following lemma.

LEMMA 5.2. *Let $x, y \in Q_{m,n}$. Denote their associated semisimple representations by*

$$\phi = \phi_1^{a_r} \oplus \cdots \oplus \phi_r^{a_r} \quad \textit{and} \quad \nu = \nu_1^{b_1} \oplus \cdots \oplus \nu_s^{b_s}.$$

Assume that there is a polynomial $P(u_1, \ldots, u_{m-1}) \in F_{m-1}$ such that

$$\operatorname{tr} \phi(P) \neq \operatorname{tr} \nu(P).$$

Then there exists a tame automorphism f of $T_{m,n}$ such that $(f_* p, f_* q)$ is well behaved.

Proof. Consider the lower triangular automorphism f of $T_{m,n}$ given by

$$\begin{aligned} X_1 &\to X_1, \\ &\vdots \\ X_{m-1} &\to X_{m-1}, \\ X_m &\to \operatorname{tr} P(X_1, \ldots, X_{m-1}) + cX_m. \end{aligned}$$

We claim that $(f_* p, f_* q)$ is well behaved for all but finitely many values of $0 \neq c \in k$. Denote the semisimple representation associated to $f_* p$ and $f_* q$ by

$$\phi(c) = \phi_1(c)^{a_r} \oplus \cdots \oplus \phi_r(c)^{a_r} \quad \text{and} \quad \nu(c) = \nu_1(c)^{b_1} \oplus \cdots \oplus \nu_s(c)^{b_s}.$$

Note the $\phi_i(c)(u_j) = \phi_i(u_j)$ and $\nu_i(c)(u_j) = \nu_i(u_j)$ for every $j = 1 \ldots, m-1$, but

$$\phi_i(c)(u_m) = \operatorname{tr} \phi(P) I_{d_i} + c\phi_i(u_m),$$

$$\nu_i(c)(u_m) = \operatorname{tr} \nu(P) I_{d_i} + c\nu_i(u_m),$$

where I_{d_i} is the $d_i \times d_i$ identity matrix.

Choose $i = 1, \ldots, r$ and $j = 1, \ldots, s$. It is enough to show that $\phi_i(c) \not\approx \nu_j(c)$ for all but finitely many c. This follows from the fact that

$$\operatorname{tr} \phi_i(c)(u_m) = \operatorname{tr} \nu_i(c)(u_m) \tag{6}$$

for at most finitely many c. Indeed, both sides of (6) are polynomials in c. These polynomials are not identical, since equality does not hold when $c = 0$. ∎

REDUCTION 3. *We may assume without loss of generality that the restrictions of ρ and μ to $F_{m-1} = k\{u_1, \ldots, u_{m-1}\}$ coincide.*

Proof. Recall that ρ and μ are defined up to conjugation in $M_n(k)$. Denote the restrictions of ρ and μ to F_{m-1} by $\tilde{\rho}$ and $\tilde{\mu}$ respectively. Since p and q are saturated, these representations are semisimple. If $\tilde{\rho} = g\tilde{\mu}g^{-1}$ for some $g \in \mathrm{GL}_n(k)$, then we are done after replacing μ by $g\mu g^{-1}$.

We will now prove Proposition 5.1 under the assumption that no such g exists. Since $\tilde{p}$ and $\tilde{\mu}$ are semisimple, there is a polynomial $P(u_1, \dots, u_{m-1})$ such that

$$\operatorname{tr} \rho(P) \neq \operatorname{tr} \mu(P).$$

The result now follows from Lemma 5.2. ■

We are now ready to finish the proof of Proposition 5.1. Since p and q are distinct points of $Q_{m,n}$ and we are assuming that ρ and μ coincide on F_{m-1}, we must have $\rho(u_m) \neq \mu(u_m)$. Since ρ and μ are saturated, $\rho(u_m)$ and $\mu(u_m)$ lie in the semisimple subalgebra $\rho(F_{m-1}) = \mu(F_{m-1})$ of $M_n(k)$. Since the trace form on this subalgebra is nondegenerate, there exists an element $A(u_1, \dots, u_{m-1})$ such that

$$\operatorname{tr} \rho(u_m A) \neq \operatorname{tr} \mu(u_m A). \tag{7}$$

Let σ be a lower triangular automorphism of $T_{m,n}$ given by

$$\begin{aligned} u_1 &\to u_1, \\ &\vdots \\ u_{m-1} &\to u_{m-1}, \\ u_m &\to A(u_1, \dots, u_{m-1}) + cu_m. \end{aligned}$$

Note that

$$\operatorname{tr} \rho \circ \sigma\left(u_m^2\right) \neq \operatorname{tr} \mu \circ \sigma\left(u_m^2\right). \tag{8}$$

Indeed, both sides of (8) are quadratic polynomials in c. The left-hand side is equal to

$$\operatorname{tr} \rho(A^2) + 2c \operatorname{tr} \rho(u_m A) + c^2 \operatorname{tr} \rho\left(u_m^2\right),$$

and the right-hand side is

$$\operatorname{tr} \mu(A^2) + 2c \operatorname{tr} \mu(u_m A) + c^2 \operatorname{tr} \mu\left(u_m^2\right).$$

By (7) these quadratic polynomials have unequal linear terms. Hence, the inequality (8) holds for all but finitely many c, as claimed. We fix one such $c \neq 0$.

We are now ready to apply Lemma 5.2. Our polynomial $P(u_1, \ldots, u_{m-1})$ will be u_1^2. In view of (8), all we need to do now is permute the variables u_1 and u_m. More precisely, let δ be the linear automorphism of F_m given by $\delta(u_1) = u_m$, $\delta(u_m) = u_1$, and $\delta(u_i) = u_i$ for $i = 2, \ldots, m-1$. Let $x = (\sigma')_* \circ (\delta')_* p$ and $y = (\sigma')_* \circ (\delta')_* q$. The associated semisimple representations of these points are $\phi = \rho \circ \sigma \circ \delta$ and $\nu = \mu \circ \sigma \circ \delta$ respectively. We can now rewrite (8) as

$$\operatorname{tr} \phi(P) \neq \operatorname{tr} \nu(P).$$

By Lemma 5.2 there exists a tame automorphism f of $T_{m,n}$ such that $(f_* x, f_* y)$ is well behaved. In other words, $(f_* \circ (\sigma')_* \circ (\delta')_* p, f_* \circ (\sigma')_* \circ (\delta')_* q)$ is well behaved. This complete the proof of Proposition 5.1.

6. A TRIVIALITY CRITERION AND THE GURALNICK-MONTGOMERY THEOREM

In this section we prove the following proposition.

PROPOSITION 6.1. *Suppose that* $Z \in T_{m,n}$ *has the same determinant as* X_i *for some* $i = 1, \ldots, m$.

(a) *Let* $n \geqslant 3$. *Then* $Z = \zeta X_i$.
(b) *Let* $n = 2$. *Then* $Z = \zeta X_i$ *or* $Z = \zeta(\operatorname{tr} X_i - X_i)$.

Here $\zeta \in k$ *is an* n*th root of unity.*

As an immediate corollary of part (a) we obtain the following necessary and sufficient condition for $f \in \operatorname{Aut}(T_{m,n})$ to be trivial.

COROLLARY 6.2. *Suppose* f *is a* k*-algebra homomorphism of* $T_{m,n}$ *where* $n \geqslant 3$. *Then* $f = \mathrm{id}$ *if and only if for every* $i = 1, \ldots, m$ *the element* $f(X_i)$ *has the same trace and the same determinant as* X_i.

Proof. Let $Z_i = f(X_i)$. By Proposition 6.1(a) $\det Z_i = \det X_i$ implies $Z_i = \zeta X_i$. Since $\operatorname{tr} Z_i = \operatorname{tr} X_i$, we must have $\zeta = 1$. In other words, $Z_i = f(X_i) = X_i$ for every $i = 1, \ldots, m$. By Corollary 2.9 this is only possible if $f = \mathrm{id}$. ∎

In the course of proving Proposition 6.1 we shall repeatedly appeal to the following elementary lemma.

LEMMA 6.3. *Let R be a commutative k-algebra, and let*

$$A(s) = A_d s^d + A_{d-1} s^{d-1} + \cdots + A_0$$

and

$$B(s) = B_d s^d + B_{d-1} s^{d-1} + \cdots + B_0$$

for some $A_1, \ldots, A_d, B_0, \ldots, B_d \in M_n(R)$. Suppose $\det A(s) = \det B(s)$ *for every $s \in k$. Then* $\det A_d = \det B_d$.

Proof. Note that $\det A(s) = s^{jd} \det A_d + O(s^{jd-1})$ and $\det B(s) = s^{jd} \det B_d + O(s^{jd-1})$. Since the base field k is infinite, this implies $\det A_d = \det B_d$. ∎

We now recall that the trace ring $T_{m,n}$ is multigraded by the degree in $X_1, \ldots, X_m$; see Section 2. In particular, a monomial $Z = X_1^{i_1} \cdots X_m^{i_m}$ is homogeneous of multidegree $(i_1, \ldots, i_m)$, and so is $\operatorname{tr} Z$. The degree of an element $W \in T_{m,n}$ with respect to X_i will be denoted by $\deg_i(W)$.

We now proceed with the proof of Proposition 6.1. We may assume without loss of generality that $i = 1$.

LEMMA 6.4. *Suppose* $\det Z = \det X_1$. *Then* $\deg_i(Z) = 0$ *for* $i = 2, \ldots, m$, *i.e., Z is independent of $X_2, \ldots, X_m$.*

Proof. Assume the contrary, say $\deg_2(Z) = d \geqslant 1$. Write $Z = Z_d + Z_{d-1} + \cdots + Z_0$, where Z_i is a homogeneous element of degree i with respect to X_2 and $Z_d \neq 0$. Let $C = (C_1, \ldots, C_m)$ be an m-tuple of matrices from $M_n(k)$. Let

$$\begin{aligned} A(s) &= Z(C_1, sC_2, C_3, \ldots, C_n) \\ &= s^d Z_d(C) + s^{d-1} Z_{d-1}(C) + \cdots + Z_0(C). \end{aligned}$$

Applying Lemma 6.2 to $A(s)$ and $B(s) \equiv C_1$, we see that $\det Z_d(C) = 0$. Since the last equality holds for every m-tuple C of matrices from $M_n(k)$, we have $\det Z_d = 0$. By Proposition 2.4, this implies $Z_d = 0$, contradicting our assumption. ∎

LEMMA 6.5. *Suppose* $\det Z = \det X_1$. *Then* $\deg_1(Z) \leqslant 1$.

Proof. By Lemma 6.4 we may assume $\deg_i(Z) = 0$ for $i = 2, \ldots, m$, i.e., Z is independent of $X_2, \ldots, X_m$. Let $\deg_1(Z) = d$, i.e., $Z = Z_d + Z_{d-1}$

$+ \cdots + Z_1 + Z_0$, where Z_i is a homogeneous element of degree i, $Z_d \neq 0$, and $d = \deg_1(Z)$. Substituting sA for X_1, we see that sA has the same determinant as

$$Z(sA) = s^d Z_d(A) + s^{d-1} Z_{d-1}(A) + \cdots + Z_0(A).$$

We want to show $d \leqslant 1$. Assume the contrary. Then Lemma 6.2 with $A_i = Z_i(A)$, $B_1 = A$, and $B_2 = \cdots = B_d = 0$ says that $\det Z_d(A) = 0$. Moreover, this is true for every A, i.e., $\det Z_d = 0$. By Proposition 2.4, $Z_d = 0$, a contradiction. ■

We have therefore established that $Z = \alpha \operatorname{tr} X_1 + \beta X_1 + \gamma$ for some α, β, and $\gamma \in k$. Substituting the zero matrix for X_1, we obtain $\gamma^n = \det 0 = 0$, i.e. $\gamma = 0$. Thus $Z = \alpha \operatorname{tr} X_1 + \beta X_1$. Substituting the diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for X_1, we obtain a diagonal matrix

$$\operatorname{diag}(\alpha S + \beta\lambda_1, \ldots, \alpha S + \beta\lambda_n)$$

whose determinant equals $\det X_1$. Here $\lambda_1, \ldots, \lambda_n$ are independent commuting variables and $S = \lambda_1 + \cdots + \lambda_n$. In other words,

$$\prod_{i=1}^{n} (\alpha S + \beta\lambda_i) = \prod_{i=1}^{n} \lambda_i.$$

By unique factorization in $k[\lambda_1, \ldots, \lambda_n]$ there are scalars $a_1, \ldots, a_n \in k$ and a permutation $\sigma \in S_n$ such that $\alpha S + \beta\lambda_i = a_i \lambda_{\sigma(i)}$ for every $i = 1, \ldots, n$. We can now finish the proof of Proposition 6.1.

(a) $n \geqslant 3$: The equality $\alpha S = -\beta\lambda_i + a_i\lambda_{\sigma(i)}$ is impossible unless $\alpha = 0$; just set $\lambda_i = \lambda_{\sigma(i)} = 0$ to get a contradiction. On the other hand, $\alpha = 0$ implies $Z = \beta X_1$. Taking the determinant, we see that $\beta^n = 1$.

(b) $n = 2$: In this case there are only two possibilities for the permutation σ: either $\sigma = \mathrm{id}$ or $\sigma = (1, 2)$.

Suppose $\sigma = \mathrm{id}$. Taking $i = 1$, we obtain $\alpha\lambda_1 + \alpha\lambda_2 = -\beta\lambda_1 + a_1\lambda_1$ and thus $\alpha = 0$, i.e. $Z = \beta X_1$. Taking the determinant, we obtain $\beta^2 = 1$.

On the other hand, if $\sigma = (1, 2)$, then $\alpha\lambda_1 + \alpha\lambda_2 = -\beta\lambda_1 + a_1\lambda_2$ and thus $\alpha = -\beta$. Hence, $Z = \alpha(\operatorname{tr} X_1 - X_1)$. Taking the determinant, we obtain $\alpha^n = 1$, and the proof is complete.

We now derive the following theorem of Guralnick and Montgomery from Proposition 6.1. (An analogous theorem for $G_{m,n}$ was proved by Montgomery in [9]; see also Lvov and Kharchenko [8] and Le Bruyn [7, Theorem 13].)

COROLLARY 6.6 (Guralnick and Montgomery [5]). *Let f be a k-algebra automorphism of the trace ring $T_{m,n}$. Suppose $f(c) = c$ for every c of $C_{m,n}$. Then*

(a) *f is a trivial automorphism of $T_{m,n}$ if $n \geqslant 3$ or $m \geqslant 3$;*
(b) *f is either trivial or is given by $f(X_i) = \operatorname{tr} X_i - X_i$ $(i = 1, 2)$ if $n = m = 2$.*

Proof. By Proposition 2.8

$$\det f(Z) = \det Z \quad \text{and} \quad \operatorname{tr} f(Z) = \operatorname{tr} Z \tag{9}$$

for every $Z \in T_{m,n}$.

(a): If $n \geqslant 3$, (9) implies $f = \mathrm{id}$ by Corollary 6.2.

(b): If $n = 2$, then Proposition 6.1(b) implies $f(X_i) = X_i$ or $\operatorname{tr} X_i - X_i$ for each $i = 1, \ldots, m$. We claim that if the first possibility occurs for one i, then it has to occur for all other i. Indeed, suppose $f(X_1) = X_1$ but $f(X_2) = \operatorname{tr} X_2 - X_2$. Then

$$\operatorname{tr} X_1 X_2 = f(\operatorname{tr} X_1 X_2) = \operatorname{tr}\left[f(X_1) f(X_2)\right] = \operatorname{tr} X_1 \operatorname{tr} X_2 - \operatorname{tr} X_1 X_2.$$

Substituting the elementary matrices e_{11} for X_1 and e_{22} for X_2, we obtain $0 = 1$, a contradiction.

If $f(X_i) = X_i$ for some $i = 1, \ldots, m$, then $f(X_i) = X_i$ for every i, and f is trivial by Corollary 2.9. The only other possibility is $f(X_i) = \operatorname{tr} X_i - X_i$ for every $i = 1, \ldots, m$. However, if such an automorphism existed for $m \geqslant 3$, then we would have

$$\operatorname{tr} X_1 X_2 X_3 = \operatorname{tr} f(X_1 X_2 X_3) = \operatorname{tr}\left[(\operatorname{tr} X_1 - X_1)(\operatorname{tr} X_2 - X_2)(\operatorname{tr} X_3 - X_3)\right].$$

Substituting the elementary matrices e_{12} for X_1, e_{21} for X_2, and e_{11} for X_3, we obtain $1 = 0$, a contradiction. ■

7. INDUCED AUTOMORPHISMS

We now return to the group homomorphisms

$$\operatorname{Aut}(F_m) \to \operatorname{Aut}(T_{m,n}) \to \mathbf{Aut}(Q_{m,n}) \tag{10}$$

defined in Section 3. Recall that the first map is given by $\sigma \to \sigma'$ and the second map is given by $f \to f_*$.

THEOREM 7.1. *Let $m, n \geqslant 2$. In the diagram* (10),

(a) *the second map is injective iff* $(m, n) \neq (2, 2)$,
(b) *the first map is not surjective, and*
(c) *the first map is injective iff* $m = 2$.

Proof. (a) is a direct consequence of the Guralnick-Montgomery theorem 6.6.

(b): By Proposition 2.10 there is an automorphism f of $T_{m,n}$ given by

$$\begin{aligned} X_1 &\to X_1 + \operatorname{tr} X_2, \\ X_2 &\to X_2, \\ &\vdots \\ X_m &\to X_m. \end{aligned}$$

We claim that this automorphism is not induced by an automorphism of the polynomial ring F_m. Indeed, assume the contrary. Then we have an identity of the form $\operatorname{tr} X_1 = P(X_1, \ldots, X_M)$ for some polynomial $P \in F_m$. Since the left-hand side is multihomogeneous of degree $(1, 0, \ldots, 0)$, the right-hand side must be equal to αX_1 for some $\alpha \in k$, a contradiction.

(c): We first consider the case $m \geqslant 3$. Let $r \in k\{u_1, u_2\}$ be a two-variable polynomial identity for the matrix algebra $M_n(k)$. For example, $r(u_1, u_2)$ can be defined as $S_{n+1}(u_1, u_1u_2, \ldots, u_1^n u_2)$, where S_{n+1} is the standard identity of degree $n + 1$. Now let $\sigma \in \operatorname{Aut}(F_m)$ be given by

$$\begin{aligned} u_1 &\to u_1, \\ u_2 &\to u_2, \\ &\vdots \\ u_m &\to u_m + (u_1, u_2). \end{aligned}$$

Then $\sigma \neq \mathrm{id}$ in $\operatorname{Aut}(F_m)$ but $\sigma' = \mathrm{id}$ in $\operatorname{Aut}(T_{m,n})$, as desired.

Now suppose $m = 2$. Let $I_j \subset F_2$ be the ideal of polynomial identities of $j \times j$ matrices. Identifying F_m/I_n with $G_{m,n}$ and F_m/I_1 with the commutative polynomial ring $k[x_1, x_2]$, we see that the "abelianization" homomorphism $\operatorname{Aut}(F_2) \to \operatorname{Aut}(k[x_1, x_2])$ factors through $\operatorname{Aut}(G_{2,n})$. The "abelianization" homomorphism is known to be an isomorphism in the two-variable case;

see [3]. Thus the map $\mathrm{Aut}(F_2) \to \mathrm{Aut}(G_{2,n})$ is injective. By Corollary 2.9 the map $\mathrm{Aut}(F_2) \to \mathrm{Aut}(T_{2,n})$ is also injective. ∎

We now turn to the proof of Theorem 1.3, which says that the homomorphism $\mathrm{Aut}(T_{m,n}) \to \mathbf{Aut}(Q_{m,n})$ in (10) is not surjective.

Let t be the automorphism of $[M_n(k)]^m$ (as an algebraic variety) given by

$$t : (A_1, \ldots, A_m) \to (A_1^t, \ldots, A_m^t).$$

Here A_i^t is the transpose of the $n \times n$ matrix A_i. Note that for any $g \in \mathrm{PGL}_n$ and any $A = (A_1, \ldots, A_m) \in (M_n)^m$ we have $t(gA) = (g^{-1})^t(tA)$. Here gA stands for the m-tuple $(gAg^{-1}, \ldots, gA_m g^{-1})$. By the universal property of the affine quotient $Q_{m,n} = [M_n(k)]^m // \mathrm{PGL}_n$, t descends to an automorphism $t_0 : Q_{m,n} \to Q_{m,n}$. It easily follows from the definition of the representation type that $\tau(t_0(p)) = \tau(p)$ for every $p \in Q_{m,n}$. In other words, $t \in \mathbf{Aut}(Q_{m,n})$. We can now state and prove a more precise form of Theorem 1.3.

PROPOSITION 7.2. *Assume that $n \geqslant 3$ and $m \geqslant 2$. Then t_0 is not induced by an automorphism of $T_{m,n}$.*

Proof. Assume the contrary: $t_0 = f_*$ for some $f \in \mathrm{Aut}(T_{m,n})$. Then by Proposition 2.8 we have

$$c_j(f(X_i)) = t_0(c_j(X_i)) = c_j(X_i^t) = c_j(X_i)$$

for every $i = 1, \ldots, m$. Here $c_j(M)$ is the jth coefficient of the characteristic polynomial of M. In particular, $f(X_i)$ has the same trace and determinant as X_i for every $i = 1, \ldots, m$. Since $n \geqslant 3$, Corollary 6.2 says that this is only possible when $f = \mathrm{id}$.

Thus we only need to show that $t_0 \neq \mathrm{id}$. Assume the contrary: $t_0 = \mathrm{id}$. Then we must have

$$\mathrm{tr}\left(X_1 X_2 X_1^2 X_2^2\right) = \mathrm{tr}\left[X_1^t X_2^t (X_1^t)^2 (X_2^t)^2\right] = \mathrm{tr}\left(X_2^2 X_1^2 X_2 X_1\right). \quad (11)$$

We now substitute $e_{12} + e_{23}$ for X_1 and $e_{21} + ae_{32}$ for X_2 into (11). Here e_{ij} is the (i, j)th elementary matrix in $M_n(k)$ and a is an element of k. The identity (11) then yields $a = a^2$ for every $a \in k$, a contradiction. ∎

REMARKS.

(1) The last argument can be made more transparent if $m \geqslant 3$. Indeed, in this case $t_0 = \text{id}$ implies

$$\operatorname{tr} X_1 X_2 X_3 = \operatorname{tr} X_1^t X_2^t X_3^t = \operatorname{tr} X_3 X_2 X_1.$$

Substituting the elementary matrices e_{12}, e_{23}, and e_{13} for X_1, X_2, and X_3 respectively, we obtain $0 = 1$, a contradiction.

(2) For $n = 2$, t_0 is in fact induced by an automorphism of the trace ring. Let $Y_i = \operatorname{tr} X_i - X_i$ for $i = 1, \dots, m$. Note that $X_i = \operatorname{tr} Y_i - Y_i$. By Proposition 2.10 there exists an automorphism f of $T_{m,2}$ which takes X_i to Y_i. I claim that $t_0 = f_*$. Indeed, it is enough to check

$$\operatorname{tr} P(fX_1, \dots, fX_m) = \operatorname{tr} P\left(X_1^t, \dots, X_m^t\right)$$

for every polynomial $P \in F_m$. This is a consequence of the identity $M^t = g(\operatorname{tr} M - M)g^{-1}$, where $g = e_{21} - e_{12}$ and M is an arbitrary 2×2 matrix.

The automorphism $t_0 = f_* : Q_{m,2} \to Q_{m,2}$ is clearly trivial when $m = 1$. (If $m = 1$ then $t = \text{id}$ and thus $t_0 = \text{id}$ for any n.) By the Guralnick-Montgomery theorem 6.6 it is also trivial when $m = 2$ and nontrivial for all other values of m.

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